Volume and commensurability of hyperbolic 3-orbifolds

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We show a relationship between the commensurability and volume of hyperbolic 3-orbifolds. Let M_1 and M_2 be non-arithmetic orientable cusped hyperbolic 3-orbifolds. If $0<|\mathrm{vol}(M_1)-\mathrm{vol}(M_2)|<v_0/4$, then M_1 and M_2 are incommensurable, where $v_0= 1.0149 \ldots$ is the volume of a regular ideal tetrahedron.

1. Introduction

Two hyperbolic 3-orbifolds are commensurable if they have a common cover, of finite degree. Two commensurable orbifolds have necessarily commensurable volumes. However, little is known about the relationship between the volume and commensurability of hyperbolic 3-orbifolds. In this paper, we show the following theorem.

Theorem 1. *Let M_1 and M_2 be non-arithmetic orientable cusped hyperbolic 3-orbifolds. If $0<|vol(M_1)- vol(M_2)|<v_0/4$, then M_1 and M_2 are incommensurable, where $v_0 = 1.0149...$ is the volume of a regular ideal tetrahedron.*

2. Preliminaries

In [1], C. Adams has determined six cusped orientable hyperbolic 3-orbifolds of volume less than $v_0/4$. W. Neumann and A. Reid have shown that these six orbifolds are arithmetic [6]. Thus we have the following proposition.

Proposition 1. *The volume of non-arithmetic orientable cusped hyperbolic 3-orbifold is larger than or equal to $v_0/4$*

For a nonarithmetic hyperbolic orbifold or manifold, G. Margulis has shown the following theorem [3].

Theorem 2. *Let M be a non-arithmetic hyperbolic 3-orbifold. Then there is an orbifold C(M) which is finitely covered by any other manifold and orbifold in the commensurability class of M.*

3. Proof of Main Theorem

Suppose that M_1 and M_2 are commensurable. As M_1 and M_2 are non-arithmetic, they cover a common orientable orbifold C. Let $P_i : M_i \to C$ be an n_i-fold covering map. Then we get $\mathrm{vol}(Mi) = n_i\mathrm{vol}(C)$ (i = 1, 2). Since $|\mathrm{vol}(M_1) - \mathrm{vol}(M_2)|>0$, $n_1 \neq n_2$. We have $|\mathrm{vol}(M_1) - \mathrm{vol}(M_2)|=|n_1\mathrm{vol}(C) - n_2\mathrm{vol}(C)|=|n_1 - n_2|\mathrm{vol}(C)$.

By Proposition 1, $\mathrm{vol}(C) \geq v_0/4$. Therefore $|\mathrm{vol}(M_1) - \mathrm{vol}(M_2)| \geq v_0/4$. This contradicts the assumption.

4. Application

T. Marshall and G. Martin have shown that the volume of a closed orientable hyperbolic 3-orbifold is larger than or equal to v_1, where $v_1 = 0.00390\cdots$ is the covolume of the Coxeter tetrahedral group [3; 5; 3] [5]. We can prove the following theorem in the same way as the proof of Theorem 1.

Theorem 3. *Let M_1 and M_2 be non-arithmetic orientable closed hyperbolic 3-orbifolds. If $0<|vol(M_1) - vol(M_2)|<0.039$, M_1 and M_2 are incommensurable.*

Corollary 2. *Let M be an $n-$cusped hyperbolic 3-manifold. Put X(M) be a set of hyperbolic manifolds which is obtained by Dehn filling on the $i-$th cusp of M. Then X(M) contains infinitely many commensurability classes.*

(Proof of Corollary 2.) Let $M(p,q)$ be a hyperbolic manifold obtained by doing a (p,q)-Dehn surgery on the i − th cusp ($i = 1,...,n$). Then we have $\mathrm{vol}(M(p,q)) < \mathrm{vol}(M)$. Let K> 0. Then there are at most finitely many arithmetic hyperbolic 3-manifolds with volume less than K. (See Theorem 11.2. in [4].) Thus at most finitely many Dehn surgeries on M can yield arithmetic hyperbolic manifolds.

By hyperbolic Dehn surgery Theorem, $vol(M(p;q))\to vol(M)$ $(p^2+ q^2\to\infty)$. By Theorem 1 and 3, X(M) contains infinitely many commensurability classes.

Remark: The above Corollary 2 is already proved in [2].

References

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