Volume and commensurability of hyperbolic 3-orbifolds

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We show a relationship between the commensurability and volume of hyperbolic 3-orbifolds. Let $M_1$ and $M_2$ be non-arithmetic orientable cusped hyperbolic 3-orbifolds. If $0 < |\text{vol}(M_1) - \text{vol}(M_2)| < v_0 / 4$, then $M_1$ and $M_2$ are incommensurable, where $v_0 = 1.0149...$ is the volume of a regular ideal tetrahedron.

1. Introduction

Two hyperbolic 3-orbifolds are commensurable if they have a common cover, of finite degree. Two commensurable orbifolds have necessarily commensurable volumes. However, little is known about the relationship between the volume and commensurability of hyperbolic 3-orbifolds. In this paper, we show the following theorem.

Theorem 1. Let $M_1$ and $M_2$ be non-arithmetic orientable cusped hyperbolic 3-orbifolds. If $0 < |\text{vol}(M_1) - \text{vol}(M_2)| < v_0 / 4$, then $M_1$ and $M_2$ are incommensurable, where $v_0 = 1.0149...$ is the volume of a regular ideal tetrahedron.

2. Preliminaries

In [1], C. Adams has determined six cusped orientable hyperbolic 3-orbifolds of volume less than $v_0 / 4$. W. Neumann and A. Reid have shown that these six orbifolds are arithmetic [6]. Thus we have the following proposition.

Proposition 1. The volume of non-arithmetic orientable cusped hyperbolic 3-orbifold is larger than or equal to $v_0 / 4$.

For a nonarithmetic hyperbolic orbifold or manifold, G. Margulis has shown the following theorem [3].

Theorem 2. Let $M$ be a non-arithmetic hyperbolic 3-orbifold. Then there is an orbifold $C(M)$ which is finitely covered by any other manifold and orbifold in the commensurability class of $M$.

3. Proof of Main Theorem

Suppose that $M_1$ and $M_2$ are commensurable. As $M_1$ and $M_2$ are non-arithmetic, they cover a common orientable orbifold $C$. Let $P_i : M_i \rightarrow C$ be an $n_i$-fold covering map. Then we get $|\text{vol}(M_i) - \text{vol}(C)| = n_i |\text{vol}(C)|$ ($i = 1, 2$). Since $|\text{vol}(M_1) - \text{vol}(M_2)| = 0$, $n_1 = n_2$. We have $|\text{vol}(M_1) - \text{vol}(M_2)| = |\text{vol}(C) - n_1 |\text{vol}(C)| = |n_1 - n_2| |\text{vol}(C)|$.

By Proposition 1, $|\text{vol}(C)| \geq v_0 / 4$. Therefore $|\text{vol}(M_1) - \text{vol}(M_2)| \geq v_0 / 4$. This contradicts the assumption.

4. Application

T. Marshall and G. Martin have shown that the volume of a closed orientable hyperbolic 3-orbifold is larger than or equal to $v_1$, where $v_1 = 0.00390...$ is the covolume of the Coxeter tetrahedral group $[3; 5; 3]$ [5]. We can prove the following theorem in the same way as the proof of Theorem 1.

Theorem 3. Let $M_1$ and $M_2$ be non-arithmetic orientable closed hyperbolic 3-orbifolds. If $0 < |\text{vol}(M_1) - \text{vol}(M_2)| < 0.039$, $M_1$ and $M_2$ are incommensurable.

Corollary 2. Let $M$ be an $n$-cusped hyperbolic 3-manifold. Put $X(M)$ be a set of hyperbolic manifolds which is obtained by Dehn filling on the $i$-th cusp of $M$. Then $X(M)$ contains infinitely many commensurability classes.

(Proof of Corollary 2.) Let $M(p, q)$ be a hyperbolic manifold obtained by doing a $(p, q)$-Dehn surgery on the $i$-th cusp ($i = 1, ..., n$). Then we have $\text{vol}(M(p, q)) < \text{vol}(M)$. Let $K > 0$. Then there are at most finitely many arithmetic hyperbolic 3-manifolds with volume less than $K$. (See Theorem 11.2. in [4].) Thus at most finitely many Dehn surgeries on $M$ can yield arithmetic hyperbolic manifolds.

By hyperbolic Dehn surgery Theorem, $\text{vol}(M(p, q)) \rightarrow \text{vol}(M)$ ($p^2 + q^2 \rightarrow \infty$). By Theorem 1 and 3, $X(M)$ contains infinitely many commensurability classes.

Remark: The above Corollary 2 is already proved in [2].
References


