Some Notes on Prehomogeneous Vector Spaces Associated with Tame-Type Quivers

Makoto NAGURA, Shin-ichi OTANI*, and Daisuke TAKEDA†

We introduce a combinatorial approach to know whether a representation associated with a tame-type quiver is a prehomogeneous vector space. Although such prehomogeneous vector spaces have already determined by C. Ringel, we will give another proof, which suggests more concrete understanding. Our approach also gives a direct method to calculate the number of basic relative invariants.

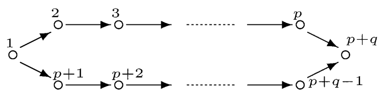
1. Introduction

A quiver is an oriented graph with a finite number of vertices. In the present notes, we will consider everything over the complex number field $\mathbb{C}$.

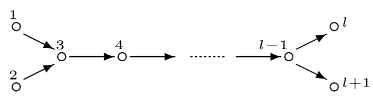
Let Q be a quiver with r vertices. We label each vertex of Q such as $1, 2, \ldots, r$ and choose an r-tuple of non-negative integers $d=(d_1,\ldots, d_r)$ (that is, d_i corresponds to the vertex i. Then the group $G_d=GL(d_1)\times\cdots\times GL(d_r)$ naturally acts on the vector space $M(d_t, d_s)$ of $d_t\times d_s$ matrices by $g\cdot x=g_t x g_s^{-1}$ for $g=(g_1,\ldots,g_r)\in G_d$ and $x\in M(d_t, d_s)$, and also on the direct sum $R_d(Q)=\bigoplus_{s\to t \text{ in } Q} M(d_t, d_s)$. In the case of $d_i=0$, we will regard as $GL(d_i)=\{1\}$, $M(d_i, d_j)=\{0\}$, and so on. Thus we have the representation $(G_d, R_d(Q))$, which is called a representation associated with the quiver Q.

A quiver Q is called to be finite-type (resp. tame-type) if its underlying graph is one of Dynkin diagrams A_l, D_l, E_6, E_7, or E_8 (resp. extended Dynkin diagrams $\widetilde{A}_l$, $\widetilde{D}_l$, $\widetilde{E}_6$, $\widetilde{E}_7$, or $\widetilde{E}_8$ (see Kraft and Riedtmann [5], §1). For example, the following one-way-oriented quivers are tame-type:

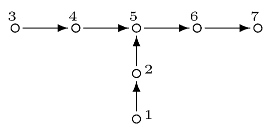
$(\widetilde{\mathbb{A}}_{p,q})$



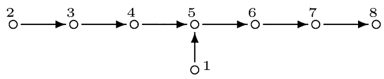
$(\widetilde{\mathbb{D}}_l)$



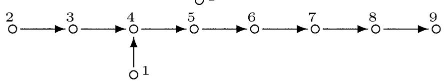
$(\widetilde{\mathbb{E}}_6)$



$(\widetilde{\mathbb{E}}_7)$



$(\widetilde{\mathbb{E}}_8)$



In fact, it is known that a quiver Q is finite-type (resp. tame-type) if and only if its corresponding quadratic form q_Q is positive definite (resp. positive semi-definite), where q_Q is defined by $q_Q(x)=\sum_{i=1}^{r} x_i^2-\sum_{s\to t \text{ in } Q} x_t x_s$ for $x={}^t(x_1,\ldots, x_r)\in\mathbb{Q}^r$ (column vectors whose entries are rational numbers).

In general, we call (G,V) a prehomogeneous vector space (abbrev. PV) if V has a Zariski dense G-orbit, where G is a linear algebraic group acting rationally on a finite-dimensional vector space V. Each point of the Zariski dense orbit is called a generic point. A non-constant rational function f on V is called a relative invariant of (G,V) if there exists a rational character χ of G satisfying $f(g\cdot v)=\chi(g)f(v)$ for any $g\in G$ and $v\in V$. Then we say that f is a relative invariant corresponding to χ. Let $X(G)$ be the group of all rational characters of G and put $X_1(G)=\{\chi\in X(G);\ \chi|_H=1\}$, where $H=\{g\in G;\ g\cdot v=v\}$ is the isotropy subgroup at a generic point v. (We call H a generic isotropy subgroup. This, of course, depends on v but it is uniquely determined up to isomorphic.) Note that $X_1(G)$ does not depend on the choice of a generic point v. Since $X(G)$ is a free abelian group, so is $X_1(G)$. In fact, it is known that if $X_1(G)=\langle\chi_1,\ldots,\chi_m\rangle$; i.e., $\chi_1,\ldots,\chi_m$ is a basis of $X_1(G)$ and if $f_1,\ldots,f_m$ are relative invariants corresponding to $\chi_1,\ldots,\chi_m$ respectively, then f_i's are algebraically independent, and moreover, any relative invariant f of (G,V) can be expressed uniquely in the

* School of Engineering, Kanto-Gakuin University
† Graduate School of Pure and Applied Sciences, University of Tsukuba

form $f = c f_1^{e_1} \cdots f_m^{e_m}$ for some $c \in \mathbb{C}^\times$ and $e_i \in \mathbb{Z}$ (see Kimura [2], §2.2). Each f_i is called a basic relative invariant. Therefore, the rank of $X_1(G)$ is nothing but the number of basic relative invariants, and to calculate it is very fundamental to the study of (G, V).

Ringel had already determined prehomogeneous vector spaces associated with tame-type quivers (see [8]), but it was somewhat difficult to know whether a representation $(G_d, R_d(Q))$ for given $d = (d_1, \dots, d_r)$ and Q is a prehomogeneous vector space. In the present notes, we give a criterion for it by a certain condition on its dimension $d = (d_1, \dots, d_r)$, and we also give an algorithm to calculate directly the number of basic relative invariants (see §5). Our strategy is based on the theory of prehomogeneous vector spaces, especially castling transformation and a certain PV-equivalent (see §3).

Note that $(G_d, R_d(Q))$ cannot be a PV if Q has a loop, because there exists a non-constant absolute invariant (see, for example, Koike [4], Theorem 1). Hence we are interested in tame-type without loop.

2. Preliminaries

Let $R_d(Q) = \bigoplus_{s \to t \text{ in } Q} M(d_t, d_s)$ be a representation associated with a quiver Q. Then a subgroup $G_1 \times \cdots \times G_r \subseteq G_d$ (here $G_i \subseteq GL(d_i)$) also acts on each component $M(d_t, d_s)$. We will write this action by $\overset{G_s}{\circ} \to \overset{G_t}{\circ}$, and simply $\overset{d_s}{\circ} \to \overset{G_t}{\circ}$ in the case of $G_s = GL(d_s)$.

Example 2.1. For example,

$$\begin{tikzcd} \overset{d_1}{\circ} \arrow[r, bend left] \arrow[r, bend right] & \overset{d_2}{\circ} \end{tikzcd}$$

means a representation of $G_d = GL(d_1) \times GL(d_2)$ on $R_d(Q) = M(d_2, d_1) \oplus M(d_2, d_1)$. We check easily that this is a PV if and only if $d_1 \neq d_2$. In the case of $d_1 = d_2$, the rational function $f(X,Y) = (\det X)/(\det Y)$ for $(X,Y) \in R_d(Q)$ is an absolute invariant. □

In general, for a subgroup $H \subseteq G_1 \times \cdots \times G_r$ and the canonical projection $\pi_i : G_1 \times \cdots \times G_r \to G_i$, we call $\pi_i(H)$ the G_i-part of H. For a partition $(e_1, e_2, \dots, e_r)$ of a positive integer n, we denote by $P(e_1, \dots, e_r)$ the corresponding standard parabolic subgroup of $GL(n)$. It is known that the $GL(d_1)$-part of the generic isotropy subgroup of $\overset{d_1}{\circ} - \cdots - \overset{d_r}{\circ}$ (which is a representation associated with an arbitrarily oriented A_r-type quiver) is expressed as a standard parabolic subgroup (see [7], Proposition 3.4).

Next we recall the so-called Coxeter matrices. Let U be a non-singular upper triangular matrix of rank r, and $A = \frac{1}{2}(U + {}^tU)$ its symmetrization. We define a symmetric bilinear form on $\mathbb{Q}^r \times \mathbb{Q}^r$ by $b(x,y) = {}^t x A y$ for column vectors $x = (x_i)$ and $y = (y_i)$ whose entries are rational numbers. For a vector α in $\mathbb{Q}^r$, the reflection σ_α is a linear transformation on $\mathbb{Q}^r$ which is defined by

$$\sigma_\alpha(x) = x - 2\frac{b(\alpha, x)}{b(\alpha, \alpha)}\alpha \quad \text{for} \quad x \in \mathbb{Q}^r.$$

For a standard column vector $e_i = {}^t(0, \dots, 1, \dots, 0)$, we write $\sigma_{e_i} = \sigma_i$ simply. We call the composite $c = \sigma_r \cdots \sigma_2 \sigma_1$ the Coxeter transformation. It is known that its representation matrix (which is called the Coxeter matrix) with respect to the standard basis $e_1, \dots, e_r$ is given by $-{}^tU^{-1}U$ (see Howlett [1], Theorem 2).

Example 2.2. For a quiver Q, we choose the upper triangular matrix U whose (i,j)-th entry is given by

$$\begin{cases} -1 & \text{if } i < j \text{ and the vertex } i \text{ is connected with } j, \\ 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Then, for a dimension vector d, we see that $b(d,d) = \dim G_d - \dim R_d(Q)$.

(1) Let $Q = \tilde{\mathbb{E}}_6$ (the one-way-oriented quiver of type $\tilde{E}_6$), which has seven vertices. Then we define the upper triangular matrix U by

$$U = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

So we can check easily that the $6n$-th power of the Coxeter matrix $C = -{}^tU^{-1}U$ is given by

$I_7 + n \cdot {}^t(1, 2, 1, 2, 3, 2, 1)(-1, -1, -1, -1, 1, 1, 1)$

$$= \begin{pmatrix} 1-n & -n & -n & -n & n & n & n \\ -2n & 1\text{-}2n & -2n & -2n & 2n & 2n & 2n \\ n & -n & 1-n & -n & n & n & n \\ -2n & -2n & -2n & 1-2n & 2n & 2n & 2n \\ -3n & -3n & -3n & -3n & 1+3n & 3n & 3n \\ -2n & -2n & -2n & -2n & 2n & 1+2n & 2n \\ -n & -n & -n & -n & n & n & 1+n \end{pmatrix},$$

where I_7 denotes the identity matrix of degree 7. The characteristic polynomial of the Coxeter matrix C is $(t^2 + t + 1)^2 (t + 1)(t - 1)^2$.

Example 2.3. Here we will calculate some power of Coxeter matrices of other tame-type quivers:

(1) The $12n$-th power of the Coxeter matrix of $\tilde{\mathbb{E}}_7$-

type is given by

$$I_8+n\cdot {}^t(2,1,2,3,4,3,2,1)(-2,-1,-1,-1,1,1,1,1).$$

The characteristic polynomial of the Coxeter matrix is $(t^2+t+1)(t^2+1)(t+1)^2(t-1)^2$.

(2) The $30n$-th power of the Coxeter matrix of $\tilde{\mathbb{E}}_8$-type is given by

$$I_9+n\cdot {}^t(3,2,4,6,5,4,3,2,1)(-3,-2,-2,1,1,\dots,1).$$

The characteristic polynomial of the Coxeter matrix is $(t^4+t^3+t^2+t+1)(t^2+t+1)(t+1)(t-1)^2$.

(3) The un-th power of the Coxeter matrix of $\tilde{\mathbb{A}}_{p,q}$-type is given by

$$I_{p+q}+n\cdot {}^t(1,\dots,1)(-w,0,\dots,0,w),$$

where u (resp v) is the least common multiple (resp. greatest common divisor) of p and q, and $w=(p+q)/v$. The characteristic polynomial of the Coxeter matrix is $(t^p-1)(t^q-1)$.

(4) The $(l-2)u$-th power of the Coxeter matrix of $\tilde{\mathbb{D}}_l$-type is given by

$$I_{l+1}+u\cdot {}^t(1,1,2,\dots,2,1,1)(-1,-1,0,\dots,0,1,1),$$

where, for $n\in\mathbb{Z}$, we put $u=2n$ if l is odd; and $u=n$ if l is even. The characteristic polynomial of the Coxeter matrix C is $(t^{l-2}-1)(t-1)(t+1)^2$.

3. PV-equivalence

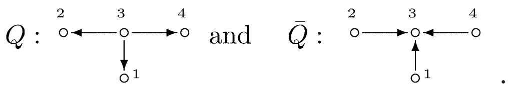
In this section, we recall two transformations among prehomogeneous vector spaces.

Proposition 3.1 ([2], Theorem 7.3 and Proposition 7.5). *The outer tensor representation $V\boxtimes V_n$ of an m-dimensional representation V of a linear algebraic group G and the standard representation of $GL(n)$ $(n<m)$ is a PV if and only if $V^*\boxtimes V_{m-n}$ is a PV. Then both generic isotropy subgroups are isomorphic, and the number of basic relative invariants of $V\boxtimes V_n$ is equal to that of $V^*\boxtimes V_{m-n}$.* □

We call $V^*\boxtimes V_{m-n}$ the castling transform of $V\boxtimes V_n$, and we say that they are castling-equivalent.

In general, a vertex s of a quiver Q is called a sink (resp. source) if all edges connecting with s are oriented toward s (resp. not oriented toward s). Sources and sinks are called admissible vertices. We can apply castling transformation to an admissible vertex if the dimensional condition is satisfied.

Example 3.2. Consider the following quivers:



For a four-tuple $d=(d_1,\dots,d_4)$ of positive integers, consider the representation $R_d(Q)$. Then we see that

$$\begin{aligned} R_d(Q)&=M(d_1,d_3)\oplus M(d_2,d_3)\oplus M(d_4,d_3)\\ &\simeq V_{d_1}^*\boxtimes V_{d_3}\oplus V_{d_2}^*\boxtimes V_{d_3}\oplus V_{d_4}^*\boxtimes V_{d_3}\\ &\simeq(V_{d_1}^*\oplus V_{d_2}^*\oplus V_{d_4}^*)\boxtimes V_{d_3}, \end{aligned}\tag{3.1}$$

where V_n is the standard representation of $GL(n)$ and V_n^* its dual. Hence, if $\bar{d}_3 := d_1+d_2+d_4-d_3>0$, the castling transform of (3.1) is given by

$$(V_{d_1}\oplus V_{d_2}\oplus V_{d_4})\boxtimes V_{\bar{d}_3},\tag{3.2}$$

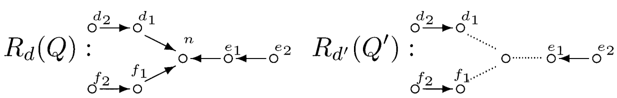
Since $GL(\bar{d}_3)$ is reductive, the representation (3.2) is isomorphic to

$$(V_{d_1}\oplus V_{d_2}\oplus V_{d_4})\boxtimes V_{\bar{d}_3}^*,\tag{3.3}$$

as PV's (see [2], §7.4). Thus we see that (3.3) is isomorphic to the representation $R_{\bar{d}}(\bar{Q})$ of dimension $\bar{d}=(d_1,d_2,\bar{d}_3,d_4)$; i.e., this is the castling transform of $R_d(Q)$. Note that the dimension $\bar{d}$ is nothing but the reflection of d with respect to the vertex 3; that is, $\bar{d}=\sigma_3(d)$. □

In a similar argument to the above, we can apply castling transformation to any admissible vertex if the dimensional condition is satisfied.

Remark 3.3. (1) Let us consider $R_d(Q)$ and $R_{d'}(Q')$:



(here $R_d(Q)$ is a representation associated with an $\tilde{E}_6$-type quiver, and $R_{d'}(Q')$ is regarded as one associated with three A_2-type quivers, which is a PV). If $d_1+e_1+f_1\leq n$ the representation $R_d(Q)$ is a PV. Moreover we see, by direct calculation, that the difference between the number of basic relative invariants of $R_d(Q)$ and that of $R_{d'}(Q')$ is given by

$$\begin{cases} 0 & \text{if } d_1+e_1+f_1<n\\ 1 & \text{if } d_1+e_1+f_1=n. \end{cases}$$

So we will say that $R_d(Q)$ can be obtained from $R_{d'}(Q')$. In fact, if the dimension (corresponding to an admissible vertex) of a representation associated with a tame-type quiver satisfies such a condition, it must be a PV because it can be regarded as a representation associated with *some* finite-type quivers.

(2) Hence, for a representation $R_d(Q)$, there exists an admissible sequence $i_1, i_2, \ldots, i_p$ such that $\sigma_{i_p} \cdots \sigma_{i_2} \sigma_{i_1}(d) \not> 0$, then it is a PV, where we say that $i_1, i_2, \ldots, i_p$ is an admissible sequence if we can apply castling transformation to vertices $i_1, i_2, \ldots, i_p$ successively.

Lemma 3.4. Let Q be a quiver of type $\tilde{A}_1$ *without loop (resp.* $\tilde{D}_l, \tilde{E}_6, \tilde{E}_7$, or $\tilde{E}_8$*), and assume that the representation* $(G_d, R_d(Q))$ *is not a PV. Then it is castling-equivalent to a representation associated with the one-way-oriented quiver* $\tilde{\mathbb{A}}_{p,q}$ *(resp.* $\tilde{\mathbb{D}}_l, \tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7$, or $\tilde{\mathbb{E}}_8$*).*

Proof. $\tilde{A}_1$-type: Let the vertex s be a sink and t a source, and assume that $s+1, s+2, \ldots, t-1$ are not admissible. First we consider the case where $s-1$ is a source as follows:

$$\text{---} \overset{s-1}{\circ} \longrightarrow \overset{s}{\circ} \longleftarrow \overset{s+1}{\circ} \longleftarrow \cdots \longleftarrow \overset{t-1}{\circ} \longleftarrow \overset{t}{\circ} \longrightarrow \cdots$$

Then, applying castling transformation to $s, s+1, \ldots, t-1$ successively, we see that $t-1$ becomes a source, and that the number of sinks and sources is reduced by just two. Next suppose that $s-1$ is not admissible. Then, applying castling transformation to $s, s+1, \ldots, t-1$ successively, we see that $s-1$ (resp. $t-1$) becomes a sink (resp. source). Thus it comes down to the first case.

$\tilde{D}_l$-type: We may assume that $1, 2, \ldots, s-1$ $(s \geq 4)$ are not admissible, and that s is a sink. Then, applying castling transformation successively, we see that $s+1$ becomes a sink. Continuing this, we obtain our assertion. The proof for $\tilde{E}_6, \tilde{E}_7$, or $\tilde{E}_8$-type is similar. Applying castling transformation to vertices from the tip of the graph, we have our assertion. ■

Proposition 3.5 ([3], Theorems 1.14 and 1.16, and Proposition 1.18) *Let W_i be an m_i-dimensional representation of a linear algebraic group G $(i=1,2)$. Assume that $n \geq \max\{m_1, m_2\}$.*

(1) *The outer tensor representation $W_1 \boxtimes W_2$ is a PV if and only if the direct sum $\widetilde{W} := W_1 \boxtimes V_n \oplus W_2 \boxtimes V_n^*$ is a PV, where V_n is the standard representation of $GL(n)$ and V_n^* its dual.*

(2) *Let l be the number of basic relative invariants of the PV $W_1 \boxtimes W_2$. Then, that of $\widetilde{W}$ is equal to $l+1$ if $n = max\{m_1, m_2\}$*; and l if $n > \max\{m_1, m_2\}$. □

In general, we say that representations (G_1, V_1) and (G_2, V_2) are PV-equivalent if the condition that (G_1, V_1) is a PV is equivalent to the one that the other is so. For example, $\widetilde{W} = W_1 \boxtimes V_n \oplus W_2 \boxtimes V_n^*$ and $W_1 \boxtimes W_2$ in Proposition 3.5 are PV-equivalent if the dimensional condition is satisfied.

Example 3.6. (1) Let us consider $R_d(Q)$ and $R_{d'}(Q')$ of dimension $d = (d_1, \ldots, d_5)$ and $d' = (d_1, d_2, d_4, d_5)$ respectively:

$$R_d(Q): \begin{tikzcd} \overset{d_1}{\circ} \arrow[dr] & & \overset{d_4}{\circ} \\ & \overset{d_3}{\circ} \arrow[ur] \arrow[dr] & \\ \overset{d_2}{\circ} \arrow[ur] & & \overset{d_5}{\circ} \end{tikzcd} \qquad R_{d'}(Q'): \begin{tikzcd} & \overset{d_4}{\circ} & \\ \overset{d_1}{\circ} \arrow[ur] \arrow[dr] & & \overset{d_2}{\circ} \arrow[ul] \arrow[dl] \\ & \overset{d_5}{\circ} & \end{tikzcd}$$

Then we see that

$$\begin{aligned} R_d(Q) &= M(d_4, d_3) \oplus M(d_5, d_3) \oplus M(d_3, d_1) \oplus M(d_3, d_2) \\ &\simeq V_{d_4}^* \boxtimes V_{d_3} \oplus V_{d_5}^* \boxtimes V_{d_3} \oplus V_{d_3}^* \boxtimes V_{d_1} \oplus V_{d_3}^* \boxtimes V_{d_2} \\ &\simeq (V_{d_4}^* \oplus V_{d_5}^*) \boxtimes V_{d_3} \oplus (V_{d_1} \oplus V_{d_2}) \boxtimes V_{d_3}^*. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} R_{d'}(Q') &\simeq V_{d_4}^* \boxtimes V_{d_1} \oplus V_{d_5}^* \boxtimes V_{d_1} \oplus V_{d_4}^* \boxtimes V_{d_2} \oplus V_{d_5}^* \boxtimes V_{d_2} \\ &\simeq (V_{d_4}^* \oplus V_{d_5}^*) \boxtimes (V_{d_1} \oplus V_{d_2}). \end{aligned}$$

Hence, if $d_3 \geq \max\{d_1 + d_2, d_4 + d_5\}$, these $R_d(Q)$ and $R_{d'}(Q')$ are PV-equivalent, so that we can know the difference between the number of basic relative invariants of the PV $R_d(Q)$ and that of $R_{d'}(Q')$.

By a similar argument to the above, if the dimension d_i corresponding to an arbitrary non-admissible vertex i satisfies such a condition, we can regard that $R_d(Q)$ and $R_{d'}(Q')$ are PV-equivalent, where $R_{d'}(Q')$ is the representation corresponding to the quiver which is obtained by removing the vertex i from Q.

4. Necessary condition to be a PV

Let us consider $R_d(\tilde{\mathbb{A}}_{p,q})$; that is, the representation, of dimension $d = (d_i)$, associated with the one-way-oriented quiver $\tilde{\mathbb{A}}_{p,q}$.

Lemma 4.1. *If the dimension d satisfies $d_1 = d_{p+q}$ and the inequality*

(4.1) $$d_1 < \min\{d_2, d_3, \dots, d_p\} + \min\{d_{p+1}, d_{p+2}, \dots, d_{p+q-1}\},$$

then $R_d(\tilde{\mathbb{A}}_{p,q})$ *is not a prehomogeneous vector space.*

Proof. Suppose that $R_d(\tilde{\mathbb{A}}_{p,q})$ is a PV, and put $d_i = \min\{d_2, \dots, d_p\}$ and $d_j = \min\{d_{p+1}, \dots, d_{p+q-1}\}$. Then $R_d(\tilde{\mathbb{A}}_{p,q})$ is PV-equivalent to the representation

(4.2)
$$\begin{tikzcd}[column sep=small, row sep=small]
 & d_{s_1} \arrow[r] & \cdots d_i \cdots \arrow[r] & d_{s_k} \arrow[dr] & \\
d_1 \arrow[ur] \arrow[dr] & & & & d_{p+q}=d_1 \\
 & d_{t_1} \arrow[r] & \cdots d_j \cdots \arrow[r] & d_{t_l} \arrow[ur] &
\end{tikzcd}$$

where $d_{s_1}, \dots, d_{s_k}$ and $d_{t_1}, \dots, d_{t_k}$ are subsequences of $d_2, \dots, d_p$ and $d_{p+1}, \dots, d_{p+q-1}$ which satisfy the inequalities $d_1 > d_{s_1} > \cdots d_i$, $d_i < \cdots < d_{s_k} < d_{p+q}$, $d_1 > d_{t_1} > \cdots > d_j$, and $d_j < \cdots < d_{r_k} < d_{p+q}$ respectively. If (4.2) is a PV, we see that the representation

(4.3)
$$\begin{tikzcd}[column sep=small, row sep=small]
 & d_i \arrow[dr] & \\
d_1 \arrow[ur] \arrow[dr] & & d_{p+q}=d_1 \\
 & d_j \arrow[ur] &
\end{tikzcd}$$

should be also a PV. Since $d_1 \leq \min\{d_i, d_j\}$, the representation (4.3) is PV-equivalent to one associated with $\tilde{\mathbb{A}}_{1,1}$ (see Example 2.1). Therefore (4.3) is not a PV; a contradiction. ■

Let us consider $R_d(\tilde{\mathbb{D}}_l)$; i.e., the representation, of dimension $d=(d_i)$, associated with the one-way-oriented quiver $\tilde{\mathbb{D}}_l$.

Lemma 4.2. *If the dimension* d *satisfies* $d_1+d_2=d_l+d_{l+1}$ *and the inequality*

$$\max\{d_1, d_2, d_l, d_{l+1}\} < \min\{d_3, d_4, \dots, d_{l-1}\},$$

then $R_d(\tilde{\mathbb{D}}_l)$ *is not a prehomogeneous vector space.*

Proof. Suppose that $R_d(\tilde{\mathbb{D}}_l)$ with such a dimension is a PV and let $d_i = \min\{d_3, \dots, d_{l-1}\}$. First we discuss the case of $d_1+d_2 = d_l + d_{l+1} \leq d_i$. Then we see that the representation

$$\begin{tikzcd}[column sep=small, row sep=small]
d_1 \arrow[dr] & & d_l \\
 & d_i \arrow[ur] \arrow[dr] & \\
d_2 \arrow[ur] & & d_{l+1}
\end{tikzcd}$$

should be a PV. This is PV-equivalent to

(4.4)
$$\begin{tikzcd}[column sep=small, row sep=small]
 & d_l & \\
d_1 \arrow[ur] \arrow[dr] & & d_2 \arrow[ul] \arrow[dl] \\
 & d_{l+1} &
\end{tikzcd}.$$

Since the representation (4.4) is castling-equivalent to

$$\begin{tikzcd}[column sep=small, row sep=small]
 & d_1+d_2-d_l \arrow[dl] \arrow[dr] & \\
d_1 \arrow[dr] & & d_2 \\
 & d_{l+1} \arrow[ur] &
\end{tikzcd},$$

this is not a PV by Lemma 4.1; a contradiction.

Next we assume that $d_1+d_2 = d_l + d_{l+1} > d_i$. Then $R_d(\tilde{\mathbb{D}}_l)$ is PV-equivalent to the representation

(4.5)
$$\begin{tikzcd}[column sep=small, row sep=small]
d_1 \arrow[dr] & & & & & d_l \\
 & d_{t_1} \arrow[r] & d_{t_2} \arrow[r] & \cdots \arrow[r] & d_{t_p} \arrow[ur] \arrow[dr] & \\
d_2 \arrow[ur] & & & & & d_{l+1}
\end{tikzcd}$$

with $d_{t_1} > d_{t_2} > \cdots > d_i$ and $d_i < \cdots < d_{t_p}$. Therefore, if (4.5) is a PV, then so is

$$\begin{tikzcd}[column sep=small, row sep=small]
d_1 \arrow[dr] & & d_l \\
 & d_i \arrow[ur] \arrow[dr] & \\
d_2 \arrow[ur] & & d_{l+1}
\end{tikzcd},$$

which is castling-equivalent to

$$\begin{tikzcd}[column sep=small, row sep=small]
d_i-d_1 & & d_i-d_l \arrow[dl] \\
 & d_i \arrow[ul] \arrow[dl] & \\
d_i-d_2 & & d_i-d_{l+1} \arrow[ul]
\end{tikzcd}.$$

It follows from a similar argument to the first case that the representation (4.6) is not a PV; a contradiction. ■

Let us consider $R_d(\tilde{\mathbb{E}}_6)$; that is, the representation, of dimension $d=(d_1, d_2, \dots, d_7)$, associated with the one-way-oriented quiver $\tilde{\mathbb{E}}_6$.

Lemma 4.3. *If the dimension* d *satisfies* $d_1+d_2+d_3+d_4 = d_5+d_6+d_7$ *and the inequalities* (1)–(15), *then* $R_d(\tilde{\mathbb{E}}_6)$ *is not a prehomogeneous vector space.*

(1) $d_4 - d_7 > 0$
(2) $d_2 - d_7 > 0$
(3) $d_6 - d_7 > 0$
(4) $d_2 + d_3 - d_6 > 0$
(5) $d_1 + d_4 - d_6 > 0$
(6) $d_5 - d_6 > 0$
(7) $-d_3 + d_6 > 0$
(8) $-d_1 + d_6 > 0$
(9) $d_2 + d_4 - d_5 > 0$
(10) $-d_2 + d_5 > 0$
(11) $-d_4 + d_5 > 0$
(12) $d_1 + d_3 - d_7 > 0$
(13) $-d_1 + d_2 > 0$
(14) $-d_3 + d_4 > 0$
(15) $-d_1 - d_3 + d_5 > 0$

Proof. Suppose that $R_d(\tilde{\mathbb{E}}_6)$ with such a dimension is a PV. Here we consider the case of $d_1 \leq d_5 - d_4$ and $d_3 \leq d_5 - d_2$ (we can prove other cases similarly). First we note that the $GL(d_5)$-part H of the generic isotropy subgroup of

$$d_3 \longrightarrow d_4 \longrightarrow d_5 \longleftarrow d_2 \longleftarrow d_1$$

is the intersection $P(d_1, d_2-d_1, d_5-d_2) \cap {}^tP(d_5-d_4, d_4-d_3, d_3)$ of two standard parabolic subgroups, and that it is contained in the $GL(d_5)$-part of the generic isotropy subgroup of the PV

$$\begin{tikzcd} \overset{d_1}{\circ} \arrow[r] & \overset{d_5}{\circ} & \overset{d_3}{\circ} \arrow[l] \\ & \underset{d_2+d_4-d_5}{\circ} \arrow[u] & \end{tikzcd}$$

Hence, for that the representation $\overset{H}{\circ} \to \overset{d_6}{\circ} \to \overset{d_7}{\circ}$ is a PV, it is necessary that the representation

(4.7) $$\begin{tikzcd} & & & \overset{d_3}{\circ} \arrow[dl] \\ \overset{d_7}{\circ} & \overset{d_6}{\circ} \arrow[l] & \overset{d_5}{\circ} \arrow[l] & \overset{d_2+d_4-d_5}{\circ} \arrow[l] \\ & & & \underset{d_1}{\circ} \arrow[ul] \end{tikzcd}$$

is a PV. Since $(d_1+d_4)+(d_5+d_3) \leq 2d_5$ and $d_5 \geq d_6$, the representation (4.7) is PV-equivalent to

$$\begin{tikzcd} & & \overset{d_3}{\circ} \arrow[dl] \\ \overset{d_7}{\circ} & \overset{d_6}{\circ} \arrow[l] & \overset{d_2+d_4-d_5}{\circ} \arrow[l] \\ & & \underset{d_1}{\circ} \arrow[ul] \end{tikzcd},$$

which is castling-equivalent to

$$\begin{tikzcd} \overset{d_1}{\circ} \arrow[dr] & & \overset{d_7}{\circ} \\ & \overset{d_6}{\circ} \arrow[ur] \arrow[dr] & \\ \underset{d_3}{\circ} \arrow[ur] & & \underset{d_1+d_3-d_7}{\circ} \end{tikzcd}.$$

Here we note that $d_6 \geq \max\{d_1, d_3, d_7, d_1+d_3-d_7\}$. It follows from Lemma 4.2 that this is not a PV; a contradiction. ■

Let us consider $R_d(\widetilde{\mathbb{E}}_7)$; that is, the representation, of dimension $d=(d_1, d_2, \ldots, d_8)$, associated with the one-way-oriented quiver $\widetilde{\mathbb{E}}_7$.

Lemma 4.4. *If the dimension* d *satisfies* $2d_1+d_5+d_3+d_4=d_5+d_6+d_7+d_8$ *and the following inequalities* (1)–(22), *then* $R_d(\widetilde{\mathbb{E}}_7)$ *is not a PV.*

(1) $d_1-d_8>0$
(2) $d_7-d_8>0$
(3) $d_4-d_7>0$
(4) $d_6-d_7>0$
(5) $d_1+d_3-d_6>0$
(6) $d_5-d_6>0$
(7) $d_6+d_7>d_1+d_3$
(8) $d_1+d_4-d_5>0$
(9) $d_1+d_3>d_7+d_8$
(10) $d_3-d_8>0$
(11) $d_5+d_8>d_1+d_3$
(12) $d_1+d_2-d_7>0$
(13) $-d_2+d_7>0$
(14) $d_1+d_4>d_6+d_8$
(15) $-d_1+d_6>0$
(16) $d_6+d_8>d_1+d_2$
(17) $-d_4+d_5>0$
(18) $d_5+d_7>d_1+d_4$
(19) $-d_3+d_4>0$
(20) $-d_3+d_6>0$
(21) $-d_2+d_3>0$
(22) $-d_1-d_2+d_5>0$

Proof. Suppose that the representation $R_d(\widetilde{\mathbb{E}}_7)$ with such a dimension is a PV. First we consider the case of $d_3 \leq d_5-d_1$ (we can prove the other case similarly). The $GL(d_5)$-part of the generic isotropy subgroup of $\overset{d_2}{\circ} \to \overset{d_3}{\circ} \to \overset{d_4}{\circ} \to \overset{d_5}{\circ} \leftarrow \overset{d_1}{\circ}$ is contained in that of $\overset{P_1}{\circ} \to \overset{d_5}{\circ} \leftarrow \overset{P_2}{\circ}$, where $P_1=P(d_1+d_4-d_5, d_5-d_4) \subset GL(d_1)$ and $P_2=P(d_2, d_3-d_2) \subset GL(d_3)$. Hence the representation

(4.8) $$\begin{tikzcd} \overset{P_1}{\circ} \arrow[r] & \overset{d_5}{\circ} \arrow[r] \arrow[d] & \overset{d_6}{\circ} \arrow[r] & \overset{d_7}{\circ} \arrow[r] & \overset{d_8}{\circ} \\ & \underset{P_2}{\circ} & & & \end{tikzcd}$$

should be a PV. The parabolic subgroup P_1 (resp. P_2) is the $GL(d_1)$-part (resp. $GL(d_3)$-part) of the generic isotropy subgroup of the representation

$$\overset{d_1+d_4-d_5}{\circ} \longrightarrow \overset{d_1}{\circ} \qquad (\text{resp. } \overset{d_2}{\circ} \longrightarrow \overset{d_3}{\circ}).$$

Thus, since $d_1+d_3 \leq d_5$ and $d_5 \geq d_6$, the representation (4.8) is PV-equivalent to

$$\begin{tikzcd} \overset{d_2}{\circ} \arrow[r] & \overset{d_3}{\circ} \arrow[r] & \overset{d_6}{\circ} \arrow[r] & \overset{d_7}{\circ} \arrow[r] & \overset{d_8}{\circ} \\ & & \overset{d_1}{\circ} \arrow[u] & & \\ & & \underset{d_1+d_4-d_5}{\circ} \arrow[u] & & \end{tikzcd}.$$

It follows from Lemma 4.3 that this is not a PV; a contradiction. ■

Let us consider $R_d(\widetilde{\mathbb{E}}_8)$; that is, the representation, of dimension $d=(d_1, d_2, \ldots, d_9)$, associated with the one-way-oriented quiver $\widetilde{\mathbb{E}}_8$.

Lemma 4.5. *If the dimension* d *satisfies* $3d_1+2d_2+2d_3=d_4+d_5+d_6+d_7+d_8+d_9$ *and the following inequalities* (1)–(29), *then* $R_d(\widetilde{\mathbb{E}}_8)$ *is not a PV.*

(1) $d_8-d_9>0$
(2) $d_7-d_8>0$
(3) $d_6-d_7>0$
(4) $d_5-d_6>0$
(5) $d_4-d_5>0$
(6) $d_1+d_3-d_4>0$
(7) $d_3-d_9>0$
(8) $d_1-d_8>0$
(9) $d_3-d_7>0$
(10) $d_1+d_2-d_6>0$
(11) $d_1+d_3>d_5+d_9$
(12) $d_1+d_2+d_3-d_4-d_8>0$
(13) $d_1+d_2>d_7+d_9$
(14) $d_1+d_3>d_6+d_8$
(15) $d_1+d_2+d_3-d_5-d_7>0$
(16) $-d_1-d_2-d_3+d_5+d_7+d_8>0$
(17) $d_1+d_2+d_3-d_5-d_8-d_9>0$
(18) $-d_1-d_2-d_3+d_5+d_6+d_9>0$
(19) $d_1+d_2+d_3-d_6-d_7-d_9>0$
(20) $-d_1-d_2-d_3+d_4+d_7+d_9>0$
(21) $d_6+d_8>d_1+d_2$
(22) $d_5+d_7>d_1+d_3$
(23) $-d_1-d_2-d_3+d_4+d_6>0$
(24) $d_5+d_9>d_1+d_3$
(25) $d_4+d_8>d_1+d_3$
(26) $-d_2+d_7>0$
(27) $-d_1+d_6>0$
(28) $-d_3+d_5>0$
(29) $-d_1-d_2+d_4>0$

Proof. Note that the representation $R_d(\tilde{\mathbb{E}}_8)$ is castling-equivalent to

$$\begin{tikzcd} \overset{d_2}{\circ} \arrow[r] & \overset{d_3}{\circ} \arrow[r] & \overset{d_4}{\circ} \arrow[r] \arrow[d] & \overset{d_5}{\circ} \arrow[r] & \overset{d_6}{\circ} \arrow[r] & \overset{d_7}{\circ} \arrow[r] & \overset{d_8}{\circ} \arrow[r] & \overset{d_9}{\circ} \\ & & \circ\, d_4-d_1 & & & & & \end{tikzcd}.$$

Suppose that this is a PV. We consider the case of $d_7 \leq d_1$ (the other case can be proved similarly). We note that the $GL(d_4)$-part of the generic isotropy subgroup, at the standard point, of $\overset{d_4-d_1}{\circ} \leftarrow \overset{d_4}{\circ} \rightarrow \overset{d_5}{\circ} \rightarrow \overset{d_6}{\circ} \rightarrow \overset{d_7}{\circ} \rightarrow \overset{d_8}{\circ} \rightarrow \overset{d_9}{\circ}$ is contained in that of $\overset{P_1}{\circ} \leftarrow \overset{d_4}{\circ} \rightarrow \overset{P_2}{\circ}$, where $P_1={}^tP(d_9, d_8-d_9, d_7-d_8) \subset GL(d_7)$ and $P_2={}^tP(d_6-d_1, d_5-d_6, d_4-d_5) \subset GL(d_4-d_1)$. Hence necessarily the representation

(4.9)
$$\begin{tikzcd} \overset{d_2}{\circ} \arrow[r] & \overset{d_3}{\circ} \arrow[r] & \overset{d_4}{\circ} \arrow[r] \arrow[d] & \overset{P_1}{\circ} \\ & & \circ\, P_2 & \end{tikzcd}$$

should be a PV. Since $d_7+(d_4-d_1) \leq d_4$ and $d_4 \geq d_3$, the representation (4.9) is PV-equivalent to

(4.10)
$$\begin{tikzcd} \overset{d_2}{\circ} \arrow[r] & \overset{d_3}{\circ} \arrow[r] \arrow[d] & \overset{P_1}{\circ} \\ & \circ\, P_2 & \end{tikzcd}.$$

The group P_1 (resp. P_2) is the $GL(d_7)$-part (resp. $GL(d_4-d_1)$-part) of the generic isotropy subgroup, at the standard point, of $\overset{d_7}{\circ} \rightarrow \overset{d_8}{\circ} \rightarrow \overset{d_9}{\circ}$ (resp. $\overset{d_4-d_1}{\circ} \rightarrow \overset{d_5-d_1}{\circ} \rightarrow \overset{d_6-d_1}{\circ}$). Therefore the representation (4.10) is PV-equivalent to

$$\begin{tikzcd} \overset{d_6-d_1}{\circ} & \underset{d_5-d_1}{\circ} \arrow[l] & \overset{d_4-d_1}{\circ} \arrow[l] & \overset{d_3}{\circ} \arrow[l] \arrow[r] & \overset{d_7}{\circ} \arrow[r] & \overset{d_8}{\circ} \arrow[r] & \overset{d_9}{\circ} \\ & & & \circ\, d_2 \arrow[u] & & & \end{tikzcd},$$

which is castling-equivalent to

$$\begin{tikzcd} \overset{d_1+d_3-d_4}{\circ} \arrow[r] & \underset{d_1+d_3-d_5}{\circ} \arrow[r] & \overset{d_1+d_3-d_6}{\circ} \arrow[r] & \overset{d_3}{\circ} \arrow[r] & \overset{d_7}{\circ} \arrow[r] & \overset{d_8}{\circ} \arrow[r] & \overset{d_9}{\circ} \\ & & & \circ\, d_2 \arrow[u] & & & \end{tikzcd}.$$

By Lemma 4.4, this is not a PV; a contradiction. ■

5. Main Theorem

First, for each one-way-oriented tame-type quiver Q_0, we characterize dimension such that $(G_d, R_d(Q_0))$ is not a prehomogeneous vector spaces.

Proposition 5.1. *Let Q_0 be the one-way-oriented quiver of type $\tilde{\mathbb{A}}_{p,q}$ (resp. $\tilde{\mathbb{D}}_l$, $\tilde{\mathbb{E}}_6$, $\tilde{\mathbb{E}}_7$, or $\tilde{\mathbb{E}}_8$). Then, for an r-tuple $d=(d_1, \ldots, d_r)$ of positive integers, the following conditions are equivalent, where r is the number of vertices of Q_0.*

(1) *$(G_d, R_d(Q_0))$ is not a PV.*

(2) *the dimension d satisfies the equality and inequalities in Lemma* 4.1 (*resp.* 4.2, 4.3, 4.4, *or* 4.5).

(3) *$c^k(d)>0$ for any integer $k \in \mathbb{Z}$, where $c=\sigma_r \cdots \sigma_2 \sigma_1$ is the Coxeter transformation corresponding to the underlying graph of Q_0.*

(4) *$c^k(d)>0$ for $k=1, 2, \ldots, u-1$ and $c^u(d)=d$, where u is defined by*

$$\begin{cases} \text{the least common multiple of } p \text{ and } q \text{ if } Q_0=\tilde{\mathbb{A}}_{p,q} \\ l-2 \text{ for even } l\text{; and } 2(l-2) \text{ for odd } l \text{ if } Q_0=\tilde{\mathbb{D}}_l \\ 6 \text{ (resp. } 12, 30) \text{ if } Q_0=\tilde{\mathbb{E}}_6 \text{ (resp. } \tilde{\mathbb{E}}_7, \tilde{\mathbb{E}}_8). \end{cases}$$

Proof. (2) ⇒ (1) has been proved in Lemma 4.1 (resp. Lemma 4.2, 4.3, 4.4, or 4.5), and (1) ⇒ (3) is obvious as seen in Remark 3.3 (2). (3) ⇒ (4): By Examples 2.2 and 2.3, the dimension d should be contained in the eigenspace with respect to 1 of c^u. (4) ⇒ (2): The condition $c^u(d)=d$ implies the equality, and the other implies the inequalities. ■

Theorem 5.2. *Let Q be a tame-type quiver without loop. Then, for the representation $(G_d, R_d(Q))$ of dimension d, the following conditions are equivalent:*

(1) *the representation $(G_d, R_d(Q))$ is not a PV.*

(2) *For any admissible sequence $i_1, i_2, \ldots, i_p$, we have $\sigma_{i_p} \cdots \sigma_{i_2} \sigma_{i_1}(d)>0$.*

Proof. (1) ⇒ (2) is obvious. Suppose that the condition (2) holds. Then, it follows from Lemma 3.4 that we can choose an appropriate admissible sequence, so that $(G_d, R_d(Q))$ is castling equivalent a representation associated with a one-way-oriented quiver. Thus we obtain our assertion by Proposition 5.1. ■

In other words, we have characterized the PV's associated with tame-type quivers:

Corollary 5.3. *Let Q be a tame-type quiver without loop. Then, for the representation $(G_d, R_d(Q))$ of dimension d, the following conditions are equivalent:*

(1) *the representation $(G_d, R_d(Q))$ is a PV.*

(2) *there exists an admissible sequence $i_1, i_2, \ldots, i_p$, such that $\sigma_{i_p} \cdots \sigma_{i_2} \sigma_{i_1}(d) \not> 0$.* □

In particular, we may say that a prehomogeneous vector space associated with a tame-type quiver can be obtained from a representation associated with *some* finite-type quivers.

Remark 5.4. (1) Let $(G_d, R_d(Q))$ be a representation associated with a tame-type quiver Q. We determine whether it is a PV as follows: First we try to make arrows to be one-way-oriented as in the proof of Lemma 3.4; that is, we apply castling transformation successively to an appropriate vertices $i_1, i_2, \ldots, i_p$. If $\sigma_{i_k} \cdots \sigma_{i_2} \sigma_{i_1}(d) \not> 0$, as seen in Remark 3.3, the representation $(G_d, R_d(Q))$ can be obtained from a representation associated with *some* finite-type quivers, so that it is a PV. If not, it is castling equivalent to a representation associated with a one-way-oriented quiver. Hence, we check easily by Proposition 5.1 whether it is a PV.

Anyway, if $(G_d, R_d(Q))$ is a PV, then it can be obtained from a representation associated with *some* finite-type quivers. It is known that the number of basic relative invariants of such a PV can be calculated directly by the orientation of arrows and its dimension (see [6]). Thus we know how many relative invariants the PV $(G_d, R_d(Q))$ have.

(2) The inequality (4.1) guarantees that there exist more than one non-constant relative invariants corresponding a common character. In other words, there exists a non-constant absolute invariant, which were obtained by Koike [4].

(3) In our strategy to determine whether a given $(G_d, R_d(Q))$ is a PV, it is essential that some power of Coxeter matrix can be calculated easily. Indeed, each characteristic polynomial of Coxeter transformation of tame-type (i.e., extended Dynkin diagram) is a product of some cyclotomic polynomials (recall Examples 2.2 and 2.3). In general, it is known that for a Coxeter matrix its characteristic polynomial is a product of some cyclotomic polynomials if and only if it is weakly periodic (see M. Sato [9]).

References

[1] R.B. Howlett, *Coxeter groups and M-matrices*, Bull. London Math. Soc. **14** (1982), no. 2, 137-141.

[2] T. Kimura, *Introduction to prehomogeneous vector spaces*, Translations of Mathematical Monographs **215**, American Mathematical Society, 2003, Translated from the 1998 Japanese original by Makoto Nagura and Tsuyoshi Niitani and revised by the author.

[3] T. Kimura, S. Kasai, M. Taguchi, and M. Inuzuka, *Some P.V.-equivalences and a classification of* 2-*simple prehomogeneous vector spaces of type* II, Trans. Amer. Math. Soc. **308** (1988), no. 2, 433-494.

[4] K. Koike, *Relative invariants of the polynomial rings over type* A_r, $\tilde{A}_r$ *quivers*, Adv. Math. **86** (1991), no. 2, 235-262.

[5] H. Kraft and Ch. Riedtmann, *Geometry of representations of quivers*, Representations of algebras (Durham, 1985), London Math. Soc. Lecture Note Ser. **116**, Cambridge Univ. Press, Cambridge, 1986, pp. 109-145.

[6] M. Nagura, *A reduction theory of representations associated with finite type quivers*, Ph.D. thesis, University of Tsukuba, 2004.

[7] M. Nagura and T. Niitani, *Conditions on a finite number* of orbits for A_r-type quivers, J. Algebra **274** (2004), no. 2, 429-445.

[8] C.M. Ringel, *The rational invariants of the tame quivers,* Invent. Math. **58** (1980), no. 3, 217-239.

[9] M. Sato, *Periodic Coxeter matrices and their associated quadratic forms*, Linear Algebra Appl. **406** (2005), 99-108.