Application of Transitive Graph for Education Support System

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This paper reports an education support system to take required credits from entrance to graduation in a college. This system is applied by a directed graph that subjects are vertices and relations among subjects are edges. We present an st-digraph that entrance and graduation are vertex s and vertex t respectively and subjects from the entrance to the graduation are vertices. These relations among subjects are transitive by adding some edges. Then, if there are minimal st-separators of the st-digraph, those are the required credits from entrance to graduation. We propose a method to decide the required credits for graduation.

KEYWORD: education support system, transitive graph, comparability graph, st-digraph, st-separator

1. Introduction

There are actually a require subject and an optional subject, etc., and it is the very severe one to study all the subjects though the students have to study a lot of subjects from entrance to graduation in an academic education. And for the educated side, it is true to want the students to master a lot of subjects if possible. Especially there are many subjects related to among each subject. If the students do not master them to some degree, obtaining deep expertise is difficult and they cannot graduate from the special subject in the professional education. Thus, the system that is the master of the credit of each subject and going on to school is a nonreversible system if student dose not get neither repeating a year nor mastering units. Here, we propose a directed graph by showing each subject about the credit earning system from entrance to graduation by vertices, and showing the relations among each subject by the arrow. In addition, we show as the directed st-graph where entrance is vertex s and graduation is vertex t, and we propose the method for the decision of the required subject for the credits toward graduation acquisition.

On the other hand, the generation problem concerning the graph is to find all subgraphs with a special character with the given graph, and the some kinds are [1]-[9] that has been considered about various classes in the graph. Taki, Masuda and Kashiwahara [8] thought about equivalence by a structural set with graph G and another structural set in other graph H same as the family of the vertex set.

This time, it was theoretically shown here to be able to apply transitive character in the graph to an educational system. Explanation of Chapter 2 defines several terms and notations. Chapter 3 describes the relation between transitive graph and the directed st-graph. In Chapter 4, it is described that the method for the decision of the required subject for the credits toward graduation acquisition is minimal st-separator in the directed st-graph, and describes the conclusion in Chapter 5.

2. Preliminaries

Here, to show the relation for the subjects from entrance to graduation by vertices and edges, the term and the definition concerning the graph are described. As described in the foregoing paragraph, entrance and graduation are shown a vertex s and a vertex t, respectively. And when the arbitrary vertices show the subjects from entrance to graduation, the relations among the subjects are shown by edges. Then the acquisition subject relations from entrance to graduation can be graphed.

Here, a directed graph is called a digraph that is distinct from undirected graph, and just a graph is as both types. An undirected edge between vertices u and v is denoted by (u, v). A directed edge from u to v is denoted by $(u \to v)$. For a graph G, $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively, of G. Therefore, the graph is denoted by $G=(V(G), E(G))$.

In a directed graph, a directed walk is a sequence of vertices and edges $(v_1, e_1, v_2, e_2, \cdots, e_k, v_{k+1})$ such that e_i is an edge from v_i to v_{i+1} for $i=1,2, \cdots, k$. A directed walk is called a directed path if no vertex appears more than once. A directed walk is called a directed cycle if no vertex appears more than once except that the first and last vertices are the same.

In a directed graph G, if the number of edges coming into vertex v is 0, then v is called a source of G. On the other hand, if the number of edges going out of v is 0, v is called a sink of G.

Definition 1: A directed st-graph is a digraph with distinguished vertices s and t.

Definition 2: An acyclic graph is a digraph that has no directed cycle.

Definition 3: An st-path is a simple directed path starting from s and ending at t in a directed st-graph.

Definition 4: A digraph is said to be transitive if the existence of two edges $(u \to v)$ and $(v \to w)$ implies the existence of edge $(u \to w)$.

Definition 5: An undirected graph is comparability graph if the edges can be oriented so that the resultant digraph is transitive.

Definition 6: A set of vertices in a graph is called an independent vertex set if no two vertices in it are mutually adjacent.

Definition 7: An independent vertex set is called maximal if it is contained in no other independent vertex set.

For undirected graph $G=(V(G),E(G))$, a maximal independent vertex set is abbreviated to MIS(G).

MIS(G)≡{S | S is a maximal independent vertex set of G}

Definition 8: A set of vertices in a directed st-graph is called a path independent vertex set if no two vertices in it are on the same st-path, and the relation of such vertices is called the path independence.

Definition 9: A path independent vertex set which is not contained in any other path independent vertex set is called a maximal path independent vertex set (MPI, for short).

For a directed st-graph $H=(V(H),E(H))$, a maximal path independent vertex set is abbreviated to MPI(H).

MPI(H)≡{S | S is a maximal path independent vertex set of H}

Definition 10: For directed edge $(u \to w)$, it is called short

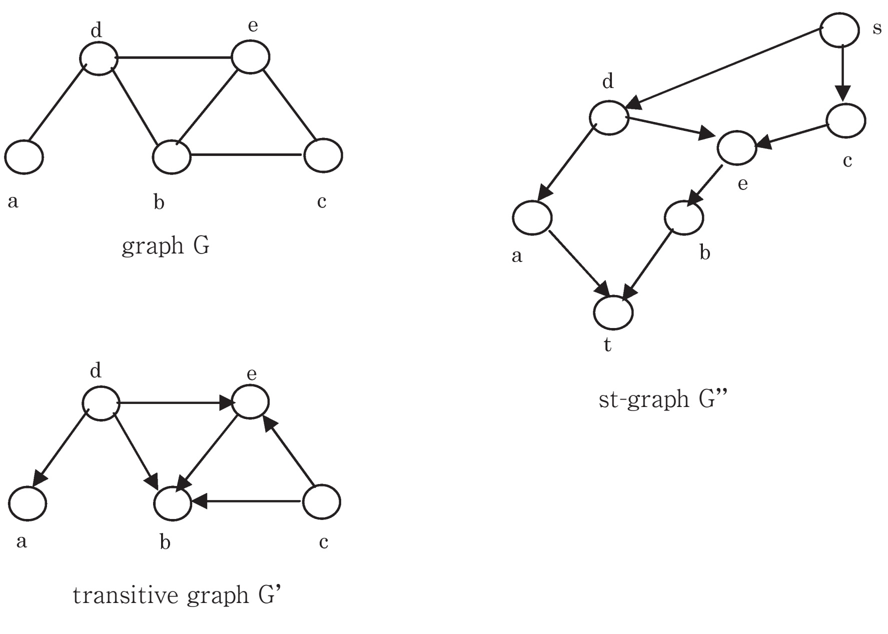

Figure 2-1 graph G, transitive graph G', and st-graph G'' from G' cut edge if there exist two directed edges $(u \rightarrow v)$ and $(v \rightarrow w)$.

Definition 11: An st-path which dose not do confluence and divergence among paths at all is called an independent st-path (Ist-path, for short). That is, Ist-paths do not share any vertices and edges.

Definition 12: An st-separator of a directed st-graph whose removal disconnects all directed paths from s to t in the st-graph is a set of vertices.

Definition 13: An st-separator is called a minimal st-separator if it has no proper set of st-separator in the st-separator.

Definition 14: An st-separator is called a minimum st-separator if it has minimum number of elements among of them.

3. Relation between a directed st-graph and a comparability graph

If a graph G is a comparability graph, the edges can be oriented so that the resultant digraph is transitive. Furthermore, such an orientation is called a transitive orientation. Let G' be the graph obtained from G by a transitive orientation.

Eliminate all short cut edges of G'. Let V_{source} and V_{sink} be the set of sources and sinks, respectively, of the resultant digraph. Create a new source s and add edge $(s \rightarrow x)$ for each $x \in V_{source}$. Similarly, create a new sink t and add edge $(y \rightarrow t)$ for each $y \in V_{sink}$. Let G'' be a st-graph obtained from G' by above procedures (figure 2-1).

Corollary 1: MPI(G'') ⊇ MIS(G)

For independent vertex a and vertex b in a graph G, it is not reachable from vertex a to vertex b and from vertex b to vertex a in the graph G'. Therefore, it is not reachable from vertex a to vertex b and from vertex b to vertex a in the graph G''. This is mean that, any two independent vertices in the graph G do not exist on same path from vertex s to vertex t in the graph G''. Thus an arbitrary element of MIS(G) is an element of a path independent vertex set of G''.

In addition, let *I1* be an arbitrary element of MIS(G). Let c be an arbitrary vertex which does not belong to *I1* in the graph G. Then, there exist an not independent at least one vertex in *I1* of vertex c. Let d be such a vertex. In the graph G', there exist a directed edge from vertex c to vertex d, or from vertex d to vertex c. Therefore, it is reachable from vertex c to vertex d or from vertex d to vertex c in the graph G''. Here, the graph G'' is made from the graph G' by deleting short cut edges of the graph G' and adding vertex s and vertex t to the graph G'. This is mean that, a reachable vertex from an arbitrary vertex in the graph G' is the same as a reachable vertex except vertex s and vertex t in the graph G''.

Therefore, there exist an st-path passing both vertex c and vertex d in the graph G''. Because this consists for arbitrary *I1* and arbitrary vertex c, MPI(G'') ⊇ MIS(G). □

Corollary 2: MPI(G'') ⊆ MIS(G)

For any two vertices a and vertex b not existing on the same st-path in a graph G'', it is not reachable from vertex a to vertex b and from vertex b to vertex a in the graph G''. Therefore, it is not reachable from vertex a to vertex b and from vertex b to vertex a in the graph G'. This is mean that, these two vertices a and b are independent in the graph G. Thus an arbitrary element of MPI(G'') is an element of a independent vertex set of the graph G.

In addition, let *MPI1* be an arbitrary element of MPI(G''). Let e be an arbitrary vertex which does not belong to *MPI1* in the graph G''. Then, there exist at least one vertex on the same st-path in *MPI1*. Let f be such an arbitrary vertex. Since it is reachable from vertex e to vertex f or from vertex f to vertex e in the graph G'', there exist a directed edge from vertex e to vertex f, or from vertex f to vertex e in the graph G'.

Therefore, there exist a directed edge between vertex e and vertex f. These two vertex e and vertex f in the graph G are not independent. Because this consists for an arbitrary vertex e and an arbitrary vertex f, MPI(G'') ⊆ MIS(G)). □

From corollary 1 and corollary 2, we obtain the following lemma.

Lemma 1: If a graph G is a comparability graph, there exist a directed acyclic st-graph with maximal path independent vertex sets that is equal to maximal independent vertex sets of G.

Because a directed st-graph is an acyclic finite graph, it has finite number of st-path with finite length. For all of st-paths, we give numbers (s → 1 → 2 → ・・・ → n → t) to vertices in the order of the sequence from vertex

s to vertex t on each st-paths. Because all of the vertices in graph H except vertex s and vertex t correspond to graph G, the vertices in graph G are numbered $1,2,\cdots,n$ in the same way. Each elements of MPI(H) dose not contain any two arbitrary vertices of the vertices, which are numbered $1,2,\cdots,n$ in the graph H, at the same time.

Corollary 3: If MPI(H)=MIS(G), a graph G can be oriented.

From MPI(H)=MIS(G), there exist undirected edges between any two arbitrary vertices which are numbered $1,2,\cdots,n$ in the graph G. When these edges are oriented toward from small number to large number which put on both ends point of each edges, it can be oriented to be transitive (Relation $<$ fills the transition for all positive integers).

For two vertices a and b which two different st-path or more in graph H passes, it is assumed that two different st-path $p1$ and $p2$ which satisfies the following conditions exist.

For numbering vertices on $p1$, (the number of vertex a) $<$ (the number of vertex b), and
for numbering vertices on $p2$, (the number of vertex a) $>$ (the number of vertex b).

At this time, the graph H has a path from vertex a to vertex b and the graph H also has a path from vertex b to vertex a. This is a contradiction to the graph H being acyclic, because the graph H has a directed cycle. Therefore, all of edges in the graph G can be oriented to be transitive. □

From corollary 3, we obtain the following lemma.

Lemma 2: If there exist a directed acyclic st-graph with maximal path independent vertex sets which is equal to maximal independent vertex sets of a graph G, the graph G is a comparability graph.

From lemma 1 and lemma 2, we obtain the following theorem.

Theorem 1: There exist a directed acyclic st-graph with maximal path independent vertex sets which is equal to maximal independent vertex sets of a graph G iff the graph G is a comparability graph.

4. St-separator in st-graph

The structure of the graph was described in the forgoing paragraph. There are some subjects of necessity at least to complete the course of study in special subject though there are various subjects in the curriculum. These are offered as requiring optional subjects according to the subject or the department. Here, various subjects (vertex v_i ($i=1,2,\cdots,k$)) from entrance (vertex s) to graduation (vertex t) are caught as directed st-graph. At this time, among the vertex sets that separate s and t, the minimal set is requiring optional subjects. That is, the method of finding the requiring optional subject is offered by catching minimal st-separator as requiring optional subjects in the directed st-graph. Moreover, student only has to find minimum st-separator from among the minimal st-separator for the master of the required subject to graduate in addition.

Here, I define the st-path degree $\rho(v_i)$ which is the number of st-path that passes vertex v_i ($i=1,2,\cdot\cdot\cdot,k$) in directed st-graph. It is the same as cutting $\rho(v_i)$ st-pathes from the directed st-graph to remove vertex v_i in the directed st-graph. Therefore, the separation of s and t becomes possible. It is the same as the exclusion of st-path from vertex s to vertex t to remove the vertex v. So, if this repeats a similar operation about st-path of the remainder, all st-path from s to t will be cut because it is limited for the number of st-path. The set of these excluded vertices becomes st-separator.

St-separator generation process:

Now, there are p st-pathes in the directed st-graph, and at this time, the maximum value of the st-path degree is Max $\rho(v_i1)=\alpha 1$. (However, the vertex with $\alpha 1$ is not necessarily one. Make any one of these vertices with some $\alpha 1$ vertex v_i1, and to same) If this vertex v_i1 is excluded, $\alpha 1$ st-pathes are excluded. When the maximum value of the st-path degree is Max $\rho(v_i2)=\alpha 2$ for $(p-\alpha 1)$ st-pathes of the remainder, $\alpha 2$ st-pathes are excluded if this vertex v_i2 is excluded. Hereafter, when the maximum value of the st-path degree is Max$\rho(v_ij)=\alpha j(j=1,2,\cdots n)$ for st-pathes of the remainder and vertex v_ij ($j=1,2,\cdots n$) is excluded, all st-path from s to t will be excluded if the similar operation is repeated. This set of vertices, which are excluded, becomes st-separator. ■

By the way, for $\alpha j(j=1,2,\cdots n)$, it is the maximum value in each operation of the above-mentioned generation process at that time, so the maximum number st-pathes are cut by removing v_ij. Therefore, removing one vertex of the maximum st-path degree in each operation was to remove maximum number st-pathes at that time, continuing this operation is following that the maximum value αj st-pathes were excluded by removing the minimum number of vertices. Obtained st-separator is unique though removed st-path is different because of how to choose the vertex with maximum st-path degree in each operation. That is, these vertex sets are minimal st-separator. The following corollary is obtained.

Corollary 4: The vertex with the maximum value of the st-path degree is an element of minimal st-separator.

From each element vertex v_j with the maximum value of the st-path degree in the above-mentioned process, $\rho(v_i)$ st-pathes are cut by excluding v_j for element v_j of minimal st-separator. That is, all st-path is separated to s and t by a minimum number of elements.

Lemma 3: The element of minimum st-separator is the vertex with the maximum value of the st-path degree in each operation of the st-separator generation process.

Now, if st-separator obtained by the above-mentioned process is not minimal st-separator, there exist st-separator as a proper subset in this st-separator. That is, there will be the vertex as st-separator besides this proper subset. By the way, because the vertex, which had been obtained for each operation by the generation process, was the vertex that the number of st-path in the directed st-graph was limited with the maximum value of the st-path degree, all st-path from s to t was cut without remaining. This is contradiction that there is the vertex as st-separator besides this proper subset. ■

The following theorem is obtained from corollary 4 and lemma 3.

Theorem 2: (maximum st-path degree - minimal st-separator theorem)

The vertex with the maximum value of the st-path degree is an element of minimal st-separator.

An actual algorithm is described based on above.

Here, st-path matrix $\mathbf{S}=[s^{ij}]$ is defined. Then

$s_{ij}=1$ if there is the vertex v_j in the st-path Pi,

$s_{ij}=0$ if there is not the vertex v_j in the st-path Pi.

The sum of the each row of the st-path matrix shows the number of st-path that passes the vertex of the row.

‘ st-path matrix **S**

‘ vm is maximum sum of row of **S**

‘ matrix **Sm** is except st-path including the maximum sum of row of the vertex

Algorithm

Start

1: S

$$v_m \leftarrow_{Max} \left(\sum v_{ij}\right)$$

Store st-spt() ← v_m :

S ← **Sm** :

If (**S**=**0**) then end

Go 1

End

5. Conclusion

Therefore, the following theorem was presented for the graph when the structure of the graph related to the subject was discussed with the start, and various subject subjects from entrance to graduation were caught as st directed graph, and the validity was proven. (1) We have shown that there exist a directed acyclic st-graph with maximal path independent vertex sets which is equal to maximal independent vertex sets of a graph G iff the graph G is a comparability graph. (2) We have also presented “Maximum st-path degree - A minimum st-separator” theorem, and proposed an actual algorithm based on it.

Each subject is made the vertex to use this for the educational support system, and the relations between subjects are connected by edges. The relations among these subjects show a transitive relation of applying the short cut edges. Then, if minimal st-separator is obtained by using the directed st-graph, the elements becomes a required subjects in educational study. A flexible to some degree a subject education is natural from the viewpoint of education though there are minimum st-separator, and minimum st-separator it, too. It is a reason for this not to obtain minimum st-separator but to obtain minimal st-separator.

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